constituted of such alternating pulses; the wave-length at which this law that intensity varies as $\lambda^4 T$ begins to assert itself affords a measure of the length on a ray over which correlation extends, viz. of the mean effective length of the constituent pulses in ordinary light.

Scrutiny of the variation of the intensity of Rubens' quartz *Reststrahlen* (1901) with temperature seems to indicate correlation extending sensibly over a range of about six wave-lengths of yellow light, for the case of these long waves at ordinary temperatures; but the formula of Planck with modern values of the constants, not specially adjusted for long waves, gives as much as eight times this range for natural radiation.

It would, of course, be possible for a graph, like the residual graph of sunspots, to be devoid of correlation over the short range that is available for scrutiny, though it would come in on a much coarser scale over a long range.]

## *The Lateral Vibrations of Bars of Variable Section.*

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(Received May 11, 1917.)

The present investigation, though strictly mathematical in character, arose in connection with a suggestion, put forward by Prof. A. Dendy and the present author in another paper communicated to the Society,* that the siliceous deposits found on certain sponge spicules occurred at nodes of the spicules, regarded as vibrating rods. These vibrations, being set up and maintained by the impact of currents of water on the spicules, are necessarily of the lateral type. For the detailed examination of such a suggestion, it is necessary to obtain a comprehensive account of the positions of the fundamental nodes on a free-free bar, as dependent on the law of variation of its cross-section. The present paper contains, in fact, the formal analysis whose results were quoted without proof in the other paper.

This analysis is of considerable generality, as will appear, and the particular examples selected for purposes of illustration, together with the manner in which the variable cross-section is dealt with, have been determined by the requirements of the biological application already mentioned. One general problem is in view throughout the work, and it may be stated as follows:—

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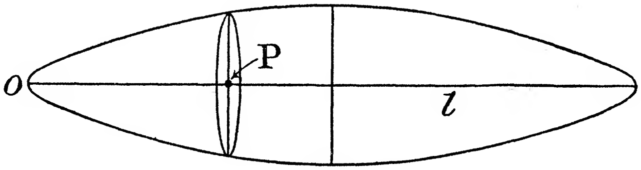
* 'Roy. Soc. Proc.,' B, vol. 89, p. 573 (1917).

If the curve $y = \mathrm{A}x^n$ revolves about the axis of $x$, forming a solid, to determine the positions of the nodes when a laterally vibrating free-free bar is formed from two equal halves fitted together at their thicker ends, each half being a section of the solid by the plane $x = l$. The bar is of length $2l$, and is shown in the figure. If the circular ring of centre P lies in a nodal plane, it is at once evident that the ratio $\mathrm{OP}/l$ is a function only of the index $n$. By the detailed discussion of suitable cases, it is possible to derive an interpolation formula giving the ratio $\mathrm{OP}/l$ as a function of $n$, from which the position of P in any practical case can be found. The results are independent of the absolute dimensions of the bar, and the range of variation of $n$ is between zero and unity.



Attention will be confined to the fundamental bi-nodal vibration, in which the two nodes are equidistant from the free ends. The ratio $\mathrm{OP}/2l$ will be called the "nodal ratio," and, in the case of a uniform bar, its value is known to be 0·2242.[*] Apart from Lord Rayleigh's discussion of the effect of a slight lack of uniformity in the cross-section,[†] which is not applicable to problems of the type now contemplated, the only detailed treatment of a bar of non-uniform section appears to be that of Kirchhoff,[‡] who has discussed the vibrations in two cases which admit of a fairly simple solution in terms of Bessel functions. One relates to a rod of prismatic section, and the other to a sharp cone, which is relevant to our present object, being a special case of the solid generated by the revolution of the curve $y = \mathrm{A}x^n$, with $n = 1$, round its axis. Kirchhoff, however, has not treated the double cone or discussed the positions of nodal planes.

When rotatory inertia is neglected—in accordance with the well-known discussion of its effect and degree of importance given by Lord Rayleigh[§]—and when the displacements are retained only to the first order after the usual manner of small oscillations, the equation of motion may be obtained very simply in the usual form,

$$\partial^2\mathrm{M}/\partial x^2 + \omega\rho \,.\, \partial^2 y/\partial t^2 = 0,$$

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[*] Lord Rayleigh, 'Theory of Sound,' vol. 1, p. 256.

[†] *Loc. cit.*, p. 264.

[‡] 'Sitz. der Akad. zu Berlin,' 1879, p. 815.

[§] *Loc. cit.*, p. 289.

where $\rho$ is the density of the material, $\omega$ its cross-section at distance $x$ from the origin, taken for convenience at a free end, and M the bending moment, given by

$$\mathrm{M} = \mathrm{E}\omega\kappa^2 \partial^2 y/\partial x^2.$$

The displacemement at any point $x$ is $y$, E is the modulus of elasticity, and $\kappa$ the radius of gyration of the area of cross-section about an axis in its plane through its centroid and perpendicular to that of the bar. We only consider circular sections, for which $\kappa^2 = \frac{1}{4}r^2$, where $r$ is the radius at the point $x$, or

$$r = \mathrm{A}x^n.$$

Thus $$\omega = \pi r^2 = \pi \mathrm{A}^2 x^{2n}, \qquad \kappa^2 = \tfrac{1}{4}r^2 = \tfrac{1}{4}\mathrm{A}^2 x^{2n},$$

and the equation of motion becomes

$$\frac{\partial^2}{\partial x^2}\left(\tfrac{1}{4}\pi \mathrm{E}\mathrm{A}^2 x^{4n}\frac{\partial^2 y}{\partial x^2}\right) + \pi\rho \mathrm{A}^2 x^{2n}\frac{\partial^2 y}{\partial t^2} = 0.$$

For a vibration of period $2\pi/p$,

$$\partial^2 y/\partial t^2 = -p^2 y,$$

and therefore $$\frac{\partial^2}{\partial x^2}\left(x^{4n}\frac{\partial^2 y}{\partial x^2}\right) = \alpha x^{2n} y,$$

where $$\alpha = 4\rho p^2/\mathrm{E}\mathrm{A}^2.$$

If $$z = qx, \qquad q^{2n-4} = \mathrm{E}\mathrm{A}^2/4\rho p^2 \quad \text{(say)}, \tag{1}$$

where $z$ is a new variable, we find

$$\frac{\partial^2}{\partial z^2}\left(z^{4n}\frac{\partial^2 y}{\partial z^2}\right) = z^{2n} y, \tag{2}$$

as the characteristic differential equation. It is convenient to discuss the special case of a double conical rod before proceeding to its general treatment.

*The Conical Rod.*—In this case, $n = 1$, and it is possible to write the differential equation in the form

$$\left(\frac{1}{z^2}\frac{\partial}{\partial z}z^3\frac{\partial}{\partial z}\right)^2 y = y,$$

so that solutions of the equations

$$\frac{1}{z^2}\frac{\partial}{\partial z}z^3\frac{\partial y}{\partial z} = \pm y \tag{3}$$

are solutions of the equation of the fourth order. This reduction to two equations of the second order is not, however, possible for any value of $n$. The new equations become

$$z\partial^2 y/\partial z^2 + 3\partial y/\partial z = \pm y.$$

When the lower sign is taken in the ambiguity, the general solution becomes

$$y = z^{-1}\{\mathrm{A}\mathrm{J}_2(2\sqrt{z}) + \mathrm{B}\mathrm{Y}_2(2\sqrt{z})\},$$

where $J_2$ is the Bessel function of order 2, and $Y_2$ the corresponding logarithmic function. With the upper sign of the ambiguity,

$$y = z^{-1}\{CI_2(2\sqrt{z})+DK_2(2\sqrt{z})\},$$

the functions being now the so-called "Bessel functions of imaginary argument," and $K_2$ being logarithmic. The general solution of the fundamental equation is

$$y = z^{-1}\{AJ_2(2\sqrt{z})+BY_2(2\sqrt{z})+CI_2(2\sqrt{z})+DK_2(2\sqrt{z})\}, \tag{4}$$

where A, B, C, D are constants determined by the conditions of support of the bar.

If the end $z = 0$ is free, $B = D = 0$, so that the displacement may not be infinite like $\log x$. The shearing stress and bending moment are respectively proportional to

$$z^4\frac{\partial^2 y}{\partial z^2}, \quad \frac{\partial}{\partial z}\left(z^4\frac{\partial^2 y}{\partial z^2}\right), \tag{5}$$

and vanish with $z$ under the same circumstances. Thus when $z = 0$ is a free end

$$y = z^{-1}\{AJ_2(2\sqrt{z})+CI_2(2\sqrt{z})\}. \tag{6}$$

As stated already, we only consider, in detail, the bar made up of two precisely equal ends. If its length is $2l$, the other end is $x = 2l$, or

$$z = x(4\rho p^2/EA^2)^{\frac{1}{2}} = 2l(4\rho p^2/EA^2)^{\frac{1}{2}} = 2\beta \quad \text{(say)}. \tag{7}$$

When the bar is vibrating symmetrically in the bi-nodal vibration, the displacements at the points $x$ and $2l-x$, or $z$ and $2\beta-z$, are equal. Thus, for the bar between $x = l$ and $x = 2l$,

$$y(2\beta-z) = AJ_2\{2\sqrt{(2\beta-z)}\}+CI_2\{2\sqrt{(2\beta-z)}\}, \tag{8}$$

the centre of the bar corresponding to $z = \beta$.

The displacement $y$ and its derivative, as given by (6) and (8), must be continuous at $x = l$ or $z = \beta$. Moreover, by symmetry, $\partial y/\partial z = 0$, and the shearing stress is zero, at the same position. These conditions are not independent.

When $z = \beta$, $y$ is evidently continuous from (6) and (8). Since

$$\frac{d}{dz}\cdot z^{-1}J_2(2\sqrt{z}) = -z^{-3/2}J_3(2\sqrt{z}), \quad \frac{d}{dz}\cdot I_2(2\sqrt{z}) = z^{-3/2}I_3(2\sqrt{z}),$$

the condition $dy/dz = 0$ at $z = \beta$ yields

$$AJ_3(2\sqrt{\beta})-CI_3(2\sqrt{\beta}) = 0. \tag{9}$$

The same condition follows from the continuity of $dy/dz$, for its values are equal and opposite on either side of $z = \beta$, and therefore zero; $d^2y/dz^2$ is continuous at $z = \beta$ automatically by (6) and (8), since

$$\frac{d^2}{dz^2} z^{-1} J_2(2\sqrt{z}) = z^{-2} J_4(2\sqrt{z}), \qquad \frac{d^2}{dz^2} z^{-1} I_2(2\sqrt{z}) = z^{-2} I_4(2\sqrt{z}).$$

The final condition is that obtained from the absence of shearing stress at $z = \beta$, or

$$\frac{\partial}{\partial z} \cdot z^4 \frac{\partial^2 y}{\partial z^2} = 0,$$

which becomes

$$\frac{d}{dz} \cdot z^2 \left\{ AJ_4(2\sqrt{z}) + CI_4(2\sqrt{z}) \right\} = 0,$$

or

$$AJ_3(2\sqrt{\beta}) + CI_3(2\sqrt{\beta}) = 0. \tag{10}$$

Comparing (9) and (10), it is evident that $C = 0$, and that the period equation is

$$J_3(2\sqrt{\beta}) = 0. \tag{11}$$

The functions of Type I thus disappear for a double conical bar vibrating symmetrically, and the period equation is simpler than in any of the cases treated by Kirchhoff.

The displacement becomes

$$y = Az^{-1}J_2(2\sqrt{z})$$

and the nodes occur where

$$J_2(2\sqrt{z}) = 0. \tag{12}$$

With $\qquad z = qx = x(4\rho p^2/EA^2)^{\frac{1}{2}} \qquad$ as in (1)

the nodes are at

$$J_2\{2\sqrt{(qx)}\} = 0, \quad \text{where} \quad J_3\{2\sqrt{(ql)}\} = 0,$$

the length of the rod being $2l$.

For the fundamental vibration, quoting the known values of the roots of these equations,*

$$2\sqrt{(ql)} = 6{\cdot}379, \qquad 2\sqrt{(qx)} = 5{\cdot}135, \tag{13}$$

and the nodal ratio is $x/2l = 0{\cdot}3240$.

In the second tone,

$$2\sqrt{(ql)} = 9{\cdot}760,$$
$$2\sqrt{(qx)} = 5{\cdot}135,\ 8{\cdot}417, \tag{14}$$

the ensuing nodal ratios being 0·1384, 0·3769.

For the third,

$$2\sqrt{(ql)} = 13{\cdot}017,$$
$$2\sqrt{(qx)} = 5{\cdot}135,\ 8{\cdot}417,\ 11{\cdot}620, \tag{15}$$

and the three nodal ratios are 0·0778, 0·20905, 0·3984.

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* Bourget, 'Ann. de l'École Normale,' vol. 3 (1866).

The three gravest notes emitted in the symmetrical vibrations are, since in the formula for $q$, $A = d/l$, where $d$ is the radius of the thickest cross-section,

$$p^2 = \frac{Ed^2q^2}{4\rho l^2} = \frac{Ed^2}{64\rho l^4}(6{\cdot}379,\ 9{\cdot}760,\ 13{\cdot}017)^4$$

and their periods are

$$\frac{2\pi}{p} = 16\pi\frac{l^2}{d}\cdot\sqrt{\frac{\rho}{E}}\cdot(6{\cdot}379,\ 9{\cdot}760,\ 13{\cdot}017)^{-2}. \tag{16}$$

*The General Period Equation.*—Whatever the value of $n$ in (2) within certain limits, it usually occurs that the equation has two solutions inapplicable, on account of an infinity, to a free end at $x = 0$. For a free-free bar consisting of two similar halves, the appropriate solution is therefore of the form

$$y = A\phi(x) + B\psi(x), \tag{17}$$

where A and B are two arbitrary constants. In the case of symmetrical vibrations, $\partial y/\partial z = 0$ and

$$\frac{\partial}{\partial z}\cdot z^{4n}\frac{\partial^2 y}{\partial z^2} = 0$$

at the centre. Thus

$$A\phi'(z) + B\psi'(z) = 0,$$

$$A\frac{d}{dz}\cdot z^{4n}\frac{d^2\phi}{dz^2} + B\frac{d}{dz}\cdot z^{4n}\frac{d^2\psi}{dz^2} = 0,$$

when $z = \beta$, the central value. Eliminating A and B, the period equation becomes, if $P = \phi(z)$, $Q = \psi(z)$,

$$P'\frac{d}{dz}\cdot z^{4n}Q'' - Q'\frac{d}{dz}\cdot z^{4n}P'' = 0, \tag{18}$$

the accent denoting differentiation to $z$, with $z = \beta$.

When P and Q are expressed in series, it is necessary to express the left side of this equation also as a series, and its general term can always be found by use of a differential equation, of higher order, which we may call a "differential period equation." This will now be obtained.

If in (2) we write $z = e^\theta$, $D \equiv d/d\theta$, it becomes, in terms of this new variable $\theta$ or $\log z$,

$$D(D-1)(D+4n-2)(D+4n-3)y = ye^{(4-2n)\theta}. \tag{19}$$

Let P and Q be any two solutions. Then

$$P\frac{d}{dz}\cdot z^{4n}Q'' - Q\frac{d}{dz}\cdot z^{4n}P''$$

$$= \{PD(D-1)(D+4n-2)Q - QD(D-1)(D+4n-2)P\}e^{(4n-3)\theta}. \tag{20}$$

Let the bracket be called $u$. Then the period equation is

$$\frac{d}{d\theta}\left(ue^{(4n-3)\theta}\right) = 0, \tag{21}$$

where $e^{\theta} = \beta$. We form a differential equation for $u$.

Let $P_rQ_s$ be a symbolic representation of $D^rP \,.\, D^sQ - D^rQ \,.\, D^sP$, so that $P_rQ_r = 0$, and, for example, $D(P_rQ_{r+1}) = P_rQ_{r+2} + P_{r+1}Q_{r+1} = P_rQ_{r+2}$. Moreover, $P_rQ_s = -P_sQ_r$, and

$$u = PQ_3 + (4n-3)\,PQ_2 - (4n-2)\,PQ_1, \tag{22}$$

and we have by (19),

$$\begin{aligned} -QP_4 = PQ_4 & \\ &= PQe^{(4-2n)\theta} - (8n-6)PQ_3 - (16n^2-28n+11)PQ_2 + (4n-2)(4n-3)PQ_1 \\ &= -(8n-6)PQ_3 - (16n^2-28n+11)PQ_2 + (4n-2)(4n-3)PQ_1, \end{aligned}$$

$$-Q_2P_4 = P_2Q_4 = P_2Qe^{(4-2n)\theta} - (8n-6)\,P_2Q_3 + (4n-2)\,(4n-3)\,P_2Q_1,$$

and so on. These examples illustrate the main properties of the symbolic notation. We find

$$Du + (4n-3)\,u = P_1Q_3 + (4n-3)\,P_1Q_2, \tag{23}$$

$$(D+4n-3)^2u = (4n-2)\,P_1Q_2 + P_2Q_3 + P_1Qe^{(4-2n)\theta}, \tag{24}$$

whence, by differentiation,

$$\begin{aligned} D\,(D+4n-3)^2u = \{2P_2Q + (4-2n)\,P_1Q\}\,e^{(4-2n)\theta} - (8n-6)\,P_2Q_3 & \\ + (4n-2)\,P_1Q_3 - (4n-2)\,(4n-3)\,P_1Q_2, & \end{aligned} \tag{25}$$

and, combining the last three equations,

$$2P_2Q + (6n-2)\,P_1Q = \psi, \tag{26}$$

where
$$\psi = e^{(2n-4)\theta}\,(D+4n-3)\,(D+4n-4)\,(D+8n-5)\,u. \tag{27}$$

By further differentiation,

$$D\psi = 2P_3Q + 2P_2Q_1 + (6n-2)\,P_2Q,$$

or, eliminating $P_3Q$ and $P_2Q$ by (22) and (26),

$$4u + (2D+2n-4)\,\psi = 4P_2Q_1 + 6n\,(2n-2)\,P_1Q. \tag{28}$$

Again differentiating, and eliminating $P_3Q_1$ (or $P_1Q_3$) and $P_2Q$,

$$\begin{aligned} 8\,(2D+4n-3)\,u + \{4D^2 + (4n-8)\,D - 6n\,(2n-2)\} & \\ = -8\,(4n-3)\,P_2Q_1 - 6n\,(2n-2)\,(6n-2)\,P_1Q, & \end{aligned} \tag{29}$$

and performing the same operations again,

$$\begin{aligned} 8\,\{D^3 + (n-2)\,D^2 - 3n\,(n-1)\,D + 3n\,(n-1)\,(3n-1)\}\,\psi & \\ + 16\,\{2D^2 + (4n-3)\,D - (4n-3)\,(D+4n-3)\}\,u & \\ = 16\,(4n-3)^2P_2Q_1 + 48n\,(n-1)\,(3n-1)^2P_1Q. & \end{aligned} \tag{30}$$

Finally eliminating $P_2Q_1$ and $P_1Q$ from (28–30), and rearranging terms,

$$(D+6n-6)(D+6n-7)(D+2n-5)(D+4n-3)(D+4n-4)(D+8n-5)\,u$$
$$= -2e^{(4-2n)\theta}(D+4n-3)(2D+6n-2)\,u, \qquad (31)$$

which is the required differential equation of the sixth order, from which the period equation may be derived. Its use is not, in fact, restricted to the special forms of end-conditions with which this paper deals, and its order can be reduced in many cases.

*The Double Paraboloid.*—We proceed to a detailed calculation for a free-free rod composed of two equal halves, as in the case of the double conical rod, each half being bounded, in any plane through the axis, by the parabola

$$y = Ax^{\frac{1}{2}}$$

between $x = 0$ and $x = 1$. Thus $n = \frac{1}{2}$, and the fundamental equation (19) becomes, since $D = zd/dz$,

$$(z\partial/\partial z)^2(z\partial/\partial z-1)^2\,y = z^3y. \qquad (32)$$

The two series solutions finite at $z = 0$ are obtained at once in the forms

$$P(z) = 1+\frac{z^3}{2^23^2}+\frac{z^6}{2^23^25^26^2}+\frac{z^9}{2^23^25^26^28^29^2}+\ldots$$

$$Q(z) = z+\frac{z^4}{3^24^2}+\frac{z^7}{3^24^26^27^2}+\frac{z^{10}}{3^24^26^27^29^210^2}+\ldots. \qquad (33)$$

Thus

$$u = P\,.\,D^2(D-1)\,Q-QD^2\,.\,D-1\,.\,P, \qquad D \equiv zd/dz \text{ by (20)},$$

begins with a term in $z^4$. The equation (31) of the sixth degree becomes

$$(D-1)^2(D-2)(D-3)(D-4)^2u = -2z^3(D-1)(2D+1)\,u, \quad D = zd/dz,$$

and its series solution, with the leading term $z^4$, is

$$u = z^4-\frac{2z^7\,.\,9}{3\,.\,4\,.\,5\,.\,6^2}+\frac{2^2z^{10}\,.\,9\,.\,15}{3\,.\,4\,.\,5\,.\,6^2\,.\,6\,.\,7\,.\,8\,.\,9^2}-$$
$$\frac{2^3z^{13}\,.\,9\,.\,15\,.\,21}{3\,.\,4\,.\,5\,.\,6^2\,.\,6\,.\,7\,.\,8\,.\,9^2\,.\,9\,.\,10\,.\,11\,.\,12^2}+\ldots. \qquad (34)$$

The period equation is, by (21),

$$\frac{d}{dz}\left(\frac{u}{z}\right) = 0, \qquad (z = \beta),$$

which takes the form

$$\sigma-\frac{2\sigma^2\,.\,4\,!}{5\,!(2\,!)^3}+\frac{3\sigma^3\,.\,6\,!}{8\,!(3\,!)^3}-\frac{4\sigma^4\,.\,8\,!}{11\,!(4\,!)^3}+\ldots = 0 \qquad (35)$$

when

$$3\sigma = \beta^3$$

This series is very convergent, as are responding series for other of the more interesting values of $n$. The first solution of the equation is readily found to be

$$\sigma = 22{\cdot}430, \qquad \beta = (3\sigma)^{\frac{1}{3}} = 4{\cdot}0674, \tag{36}$$

whence

$$\mathrm{P}' = 3\sigma^2\left\{\frac{1}{2^2\,.\,3^2}+\frac{2\sigma^3}{2^2\,.\,3^2\,.\,5^2\,.\,6^2}+\ldots\right\} = 1{\cdot}5889,$$

$$\mathrm{Q}' = 1+\frac{\sigma^3}{3^2\,.\,4}+\frac{\sigma^6}{3^2\,.\,4^2\,.\,6^2\,.\,7}+\ldots = 2{\cdot}9954, \tag{37}$$

these series being also very convergent, and suitable for calculation, and the displacement is given by the form

$$y = \mathrm{A}\left\{\frac{\mathrm{P}(z)}{1{\cdot}5889}-\frac{\mathrm{Q}(z)}{2{\cdot}9954}\right\}, \tag{38}$$

making $y'$ vanish when $z = \beta$. The node occurs where

$$\mathrm{P}(z)/\mathrm{Q}(z) = 0{\cdot}5304,$$

whose solution is $z = 2{\cdot}374.$

Thus for the fundamental vibration,

$$qx = 2{\cdot}374, \qquad ql = 4{\cdot}067$$

and the nodal ratio is $x/2l = 0{\cdot}292.$ (39)

Now by (1) $q^{4-2n} = 4\rho p^2/\mathrm{EA}^2,$

and if $d$ is the semi-breadth at the thickest part, $d = \mathrm{A}l^{\frac{1}{2}}$ defines A, and $n = \frac{1}{2}$, so that

$$q^3 = 4\rho p^2 l/\mathrm{E}d^2 = (4{\cdot}067)^3/l^3,$$

whence the period of the vibration is

$$2\pi/p = 4\pi l^2/d\,.\,\sqrt{(\rho/\mathrm{E})}\,.\,(4{\cdot}067)^{-3/2}. \tag{40}$$

*The Case* $n = 3/4$.—If $n = 3/4$, the characteristic equation (19) becomes

$$\mathrm{D}^2(\mathrm{D}-1)(\mathrm{D}+1)\,y = z^{5/2}y, \qquad \mathrm{D} \equiv z\,d/dz,$$

which admits two solutions finite at $z = 0$, in the forms

$$\mathrm{P}(z) = 1+\frac{16z^{5/2}}{3\,.\,5^2\,.\,7}+\frac{16^2\,.\,x^{10/2}}{3\,.\,5^2\,.\,7\,.\,8\,.\,10^2\,.\,12}+\frac{16^3x^{15/2}}{3\,.\,5^2\,.\,7\,.\,8\,.\,10^2\,.\,12\,.\,13\,.\,15^2\,.\,17}+\ldots,$$

$$\mathrm{Q}(z) = z^{2/2}+\frac{16z^{7/2}}{5\,.\,7^2\,.\,9}+\frac{16^2\,.\,z^{12/2}}{5\,.\,7^2\,.\,9\,.\,10\,.\,12^2\,.\,14}+\ldots. \tag{41}$$

The function

$$u = \mathrm{PD}(\mathrm{D}-1)(\mathrm{D}+1)\mathrm{Q}-\mathrm{QD}(\mathrm{D}-1)(\mathrm{D}+1)\mathrm{P}$$

has a leading term $z^{7/2}$, and satisfies the equation of the sixth orde

$$(\mathrm{D}-\tfrac{3}{2})(\mathrm{D}-\tfrac{5}{2})(\mathrm{D}-\tfrac{7}{2})\mathrm{D}(\mathrm{D}-1)(\mathrm{D}+1)\,u = 2z^{5/2}\mathrm{D}(2\mathrm{D}+\tfrac{5}{2} \qquad 42)$$

whose series solution with this leading term is, in powers of $z^{\frac{1}{2}}$,

$$u = \sum_{y}^{\infty} a_n z^{n/2}.$$

where

$$(n-2)(n+2)(n+3)(n+5)(n+7)\,a_{n+5} = -32\,(2n+5)\,a_n,$$

so that

$$u = z^{7/2} - \frac{32 z^{12/2} \,.\, 19}{5\,.\,9\,.\,10\,.\,12\,.\,14} + \frac{32^2 z^{17/2} \,.\, 19\,.\,29}{5\,.\,9\,.\,10\,.\,12\,.\,14\,.\,15\,.\,17\,.\,19\,.\,20\,.\,22\,.\,24} - \ldots \tag{43}$$

The period equation is

$$\left[\frac{d}{dz}\,.\,u z^{4n-3}\right]_{z=\beta} = 0,$$

or finally

$$0 = 7 - \frac{\sigma\,.\,12\,.\,19}{5\,.\,9\,.\,10\,.\,12\,.\,14} + \frac{\sigma^2\,.\,17\,.\,19\,.\,29}{5\,.\,9\,.\,10\,.\,12\,.\,14\,.\,15\,.\,17\,.\,19}$$

$$- \frac{\sigma^3\,.\,22\,.\,19\,.\,29\,.\,39}{5\,.\,9\,.\,10\,.\,12\,.\,14\,.\,15\,.\,17\,.\,19\,.\,20\,.\,22\,.\,24} + \ldots, \tag{44}$$

where $\sigma = 32\beta^{5/2}$.

The first root can be determined to four significant figures by including only five terms of the series. If $\sigma = 12600\tau$, we find

$$7 - 38\tau + 29\tau^2 - 7917\tau^3/1292 + +361179\tau^4/1136960 = 0,$$

whence $\tau = 0{\cdot}21966, \quad \sigma = 2767{\cdot}7, \quad \beta = 5{\cdot}9537 = ql.$ (45)

The period of the bar is then determined. The shape of one-half is given by the equation $y = \mathrm{A}x^{3/4}$, whence, if $d$ is the half-breadth at its centre,

$$\mathrm{A}^4 = d^4/l^3, \quad \text{and}$$

$$q = (4\rho p^2/\mathrm{EA}^2)^{1/(2n-4)}, \qquad n = \tfrac{3}{4},$$

$$q^5 = (16\rho^2 p^4/\mathrm{E}^2\mathrm{A}^4) = (5{\cdot}9519)^5/l^5,$$

$$= 16 p^4 \rho^2 l^3/\mathrm{E}^2 d^4,$$

so that

$$2\pi/p = 4\pi\sqrt{(\rho/\mathrm{E})}\,.\,l^2/d\,.\,(5{\cdot}9519)^{-5/4}. \tag{46}$$

The general formula is in fact

$$2\pi/p = 4\pi\sqrt{(\rho/\mathrm{E})}\,.\,l^2/d\,.\,(ql)^{n-2} \tag{47}$$

for any value of $n$.

For the values of P and Q in (41),

$$\mathrm{P}'(5{\cdot}9519) = 1{\cdot}4394, \qquad \mathrm{Q}'(5{\cdot}9519) = 3{\cdot}4612,$$

and the nodes occur where

$$\mathrm{P}(z)/1{\cdot}4394 = \mathrm{Q}(z)/3{\cdot}4612, \tag{48}$$

whose solution is $z = 3{\cdot}693$, so that the nodal ratio is $z/2l$, becoming 0·3101. The values of this ratio for $n = \frac{1}{2}$ and $n = 1$ were 0·292, 0·324, and the value for $n = \frac{3}{4}$ is almost precisely the mean of these, so that the change with $n$ is practically uniform in this region. This uniformity, however, does not extend to $n = 0$, in which case the ratio is 0·2242, and in order to obtain material for subsequent interpolation a separate investigation for $n = \frac{1}{4}$ is desirable.

*The Case* $n = \frac{1}{4}$.—For this value of $n$, the characteristic equation becomes

$$\mathrm{D}(\mathrm{D}-1)^2(\mathrm{D}-2)y = y\,.\,z^{7/2}, \qquad \mathrm{D} = z\partial/\partial z,$$

and gives three series solutions, whose leading terms are $1$, $z$, $z^2$. The fourth solution is of a logarithmic type which cannot occur with a free end at $z = 0$. The bending moment at the free end is proportional to $z\,d^2y/dz^2$ and vanishes for the form

$$y = \mathrm{A}(1+\alpha z^{7/2}+\ldots)+\mathrm{B}(z+\beta z^{9/2}+\ldots)+\mathrm{C}(z^2+\gamma z^{11/2}+\ldots)$$

derived from these three solutions. The shearing stress, which is the derivate of the bending moment, will not vanish unless $\mathrm{C} = 0$. Thus only two of the series solutions, P and Q, are required, where we find

$$\mathrm{P} = 1+\frac{16x^{7/2}}{3\,.\,5^2\,.\,7}+\frac{16^2x^{14/2}}{3\,.\,5^2\,.\,7\,.\,10\,.\,12^2\,.\,14}+\ldots,$$

$$\mathrm{Q} = x^{2/2}+\frac{16x^{9/2}}{5\,.\,7^2\,.\,9}+\frac{16^2x^{16/2}}{5\,.\,7^2\,.\,9\,.\,12\,.\,14^2\,.\,16}+\ldots. \tag{49}$$

Moreover

$$u = x^{9/2}-\frac{17\,.\,32x^{16/2}}{7^2\,.\,10^2\,.\,12}+\frac{17\,.\,31\,.\,32^2x^{23/2}}{7^2\,.\,10^2\,.\,12\,.\,14^2\,.\,17^2\,.\,19}$$
$$+\frac{17\,.\,31\,.\,45\,.\,32^3x^{30/2}}{7^2\,.\,10^2\,.\,12\,.\,14^2\,.\,17^2\,.\,19\,.\,21^2\,.\,24^2\,.\,26}+\ldots, \tag{50}$$

and the period equation is

$$\left[\frac{d}{dz}\,.\,uz^{-2}\right]_{z=\beta} = 0,$$

whose first solution is found to be

$$\beta = ql = 3{\cdot}0296, \qquad \mathrm{P}'(\beta) = 1{\cdot}8393, \qquad \mathrm{Q}'(\beta) = 2{\cdot}6387 \tag{51}$$

after calculation. The node is given by

$$\mathrm{Q}(z)/\mathrm{P}(z) = 2{\cdot}6387/1{\cdot}8393 = 1{\cdot}4346, \tag{52}$$

whence, by trial and error, $z = 1{\cdot}603 = qx$. The nodal ratio is

$$x/2l = 0{\cdot}2646, \tag{53}$$

and the fundamental vibration has a period

$$2\pi/p = 4\pi\sqrt{(\rho/\mathrm{E})}\,.\,l^2/d\,.\,(3{\cdot}0296)^{-7/4} \tag{54}$$

with the usual significance of $d$.

*Discussion of the Results.*—In the following Table, the main results worked out in detail are summarised, for the fundamental symmetrical vibration corresponding to any value of $n$ in the equation of the boundary. Those for $n = 0$ are already well known. The length of the rod is $2l$, and the greatest breadth $2b$.

| $n$. | Nodal ratio. | Frequency $\times \dfrac{d}{4\pi l^2}\sqrt{\dfrac{\text{E}}{\rho}}$. |
|---|---|---|
| 0 | 0·2242 | $(4\cdot7300)^{2/4} = 5\cdot593$ |
| 0·25 | 0·2646 | $(3\cdot0296)^{7/4} = 6\cdot957$ |
| 0·5 | 0·2920 | $(4\cdot0674)^{3/2} = 8\cdot203$ |
| 0·75 | 0·3101 | $(5\cdot9531)^{5/4} = 9\cdot300$ |
| 1·0 | 0·3240 | $(3\cdot1895)^{2} = 10\cdot173$ |

The frequency rises with the value of $n$, but not in a linear manner, and an interpolation formula in powers of $n$ between 0 and 1 is not satisfactory, though it can be used over a range of extent only 0·5. The nodes move towards the centre, as would be expected, as $n$ increases, although the nodal ratio is not quite suitable for the usual mode of interpolation except over more limited ranges. But when a great degree of accuracy is not required the interpolation formula

$$\text{Ratio} = 0\cdot074+0\cdot3\,[(4n+1)/(4n+2)] \tag{55}$$

gives good results. It may be illustrated by the spicules mentioned in the introduction, which, on the magnified drawings described in the other paper, had a length whose mean value was about 80 mm. The actual values given by the interpolation formula are:

| | | | | |
|---|---|---|---|---|
| $n = 0$ | 0·25 | 0·5 | 0·75 | 1·0 |
| Ratio $= 0\cdot224$ | 0·274 | 0·299 | 0·314 | 0·324 |
| $\Delta = 0\cdot0$ | $+0\cdot009$ | $+0\cdot007$ | $+0\cdot004$ | 0·0 |

where $\Delta$ is the error. The greatest error in nodal length on any spicule between $n = 0\cdot5$ and $n = 1\cdot0$, to which region the values of $n$ were always found to be restricted, cannot exceed $0\cdot007 \times 80$ mm., or 0·54 mm., which is within the possible error of measurement. The use of this formula in the paper is therefore justified.

It is evident that the most important determining feature in the nodal distance is the mean value of $\omega\kappa^2$ along the rod, and the factor $(4n+1)/(4n+2)$ is contained in the mean, so that the type of formula has some physical basis. The best formula which can apparently be found is

$$\text{Ratio} = 0\cdot2155-0\cdot1518\,[(4n+1)/(4n+2)]+0\cdot3384\,[(4n+1)/(4n+2)]^2 \tag{56}$$

with three constants, giving the values

| $n =$ | $0$ | $0·25$ | $0·5$ | $0·75$ | $1·00$ |
|---|---|---|---|---|---|
| Ratio $=$ | $0·2242$ | $0·2647$ | $0·2920$ | $0·3106$ | $0·3240$ |
| $\Delta =$ | $0·0$ | $+0·0001$ | $+0·0$ | $+0·0005$ | $0·0$ |

and the ratios for any values of $n$ between 0 and 1 are now known with great precision.

A formula which will give the frequencies with great accuracy is

$$\text{Frequency} = \frac{d}{4\pi l^2}\sqrt{\frac{E}{\rho}} \cdot m,$$

where
$$m = 16·237 - 0·400n(n-1)(2n-1) - 10{,}644\,(0·8688)^{4n}. \tag{57}$$

The errors in the values of $m$ thus obtained are 0·0, −0·002, 0·0, − 0·003, 0·0 respectively, corresponding to the entries for $n$ in the Table. The extent to which these interpolation formulæ can be used outside the range $n=0$ to 1 has not been investigated. In the last formula, $l$ and $d$ are the semi-length and semi-greatest breadth of the bar.

*General Considerations.*—For the purpose of numerical computation, solutions, such as that given for $n = 1$, in terms of Bessel functions are not of greater advantage. If the case $n = 1$ were treated by the present method, we should find

$$P(z) = 1 + \frac{z^2}{1.2.3.4} + \frac{z^4}{1.2.3^2.4^2.5.6} + \ldots,$$

$$Q(z) = z + \frac{z^3}{2.3.4.5} + \frac{z^5}{2.3.4^2.5^2.6.7} + \ldots = 0, \tag{58}$$

and the period equation

$$2 - \frac{3.4\beta^2}{2.5.6.8} + \frac{4.4^2\beta^4}{2.5.6.8.4.7.8.10} + \ldots = 0, \tag{59}$$

whose first solution is easily verified to be $\beta = 10·17$, in agreement with the first root of $J_3(2\sqrt{\beta}) = 0$, which is an equation in powers of $\beta$ and not of $\beta^2$. The period equation in fact contains $J_3(2\sqrt{\beta})$ as a factor, and the other factor is not oscillatory.

Kirchhoff's investigation of the conical rod appears to imply that it is the only form derived from a revolution of the curve $y \propto x^n$, which admits a solution in terms of Bessel functions. This is not in fact the case, for some other forms possess this property. Their complete classification would lead us too far from the special problem investigated in this paper, but the following may be cited as examples :—

($a$) $n = \frac{4}{5}$.—The complete solution of the equation

$$\frac{d^2}{dz^2} \cdot z^{16/5}\frac{d^2y}{dz^2} = z^{8/5}\,y$$

is
$$y = z^{1/2}\{AJ_{5/6}(\tfrac{5}{3}z^{3/5}) + BJ_{-5/6}(\tfrac{5}{3}z^{3/5}) + CI_{5/6}(\tfrac{5}{3}z^{3/5}) + DI_{-5/6}(\tfrac{5}{3}z^{3/5})\}. \tag{60}$$

(*b*) $n = 3$.—The complete solution of the equation

$$\frac{d^2}{dz^2} \cdot z^{12} \frac{d^2y}{dz^2} = z^6 y$$

is $$y = z^{-5}\{\mathrm{AJ}_{10}(2z^{-1/2}) + \mathrm{BY}_{10}(2z^{-1/2}) + \mathrm{CI}_{10}(2z^{-1/2}) + \mathrm{DK}_{10}(2z^{-1/2})\}, \quad (61)$$

where Y and K are logarithmic.

## *Summary.*

The paper contains a discussion of the lateral vibrations of a bar composed of two equal halves, and free at each end. Each half consists of a portion of the solid generated by the revolution of the curve $y = \mathrm{A}x^n$ about its axis, and the fundamental frequencies and positions of the corresponding nodes are investigated for various values of $n$ between 0 and 1.

The investigation proceeds according to a general method of determining the period equation as a series of ascending powers of a variable whose general term can be found from a differential equation of the sixth order. The special values of $n$ for which computations are made were selected in accordance with the needs of a preceding paper on the vibrations of sponge-spicules by Prof. Dendy and the writer.

The dependence of the frequency and nodal distance on the index $n$ is finally expressed by interpolation formulæ which can give an accuracy of about 1 part in 100, within the range $n = 0$ to $n = 1$.